Determining the uncertainty of measurement results with multiple measurement quantities

White Paper

By Prof. Dr.-Ing Klaus Metzger

To determine the uncertainty in the results of a single-channel measurement, one must consider the contributions from both the sensor's uncertainty and the uncertainty of the measurement channel. Results from multiple measurement channels face even greater complications. This White Paper explains the theoretical background for, as well as the practical steps taken to handle these multi-channel measurement uncertainties.

The definition of error; absolute and relative error

We begin by reviewing how error is defined. We will restrict our discussion here to systematic errors, to the exclusion of random errors. Systematic errors characteristically are of a specific size and sign. Knowledge of this error enables the measured value to be corrected. In terms of error types, there is a distinction between the absolute error and the relative error.

The (absolute) error is defined by:

$$X_F = X_A - X_W$$

(absolute error = actual reading – true value)

with X_A = value indicated and X_W = true value or expected reading. The difficulty of the matter is that a quantity's true value is typically not known. The true value can sometimes be computed, but if this is not the case, the true value often can be substituted with a value found using a reliable, high precision instrument.

For instance, an error of 1V may seem like a large amount. But it is only possible to make an assessment when the true size of the measured value is known. If it is 10V, then 1V is relatively large (10%); but if the true measurement value is 1000V, then an absolute error of 1V is relatively small (0.1%). Thus, the relative error is defined as:

$$F_r = \frac{X_F}{X_W} = \frac{X_F - X_W}{X_W} \quad \text{rel. error} = \frac{\text{actual reading–true value}}{\text{actual reading}}$$

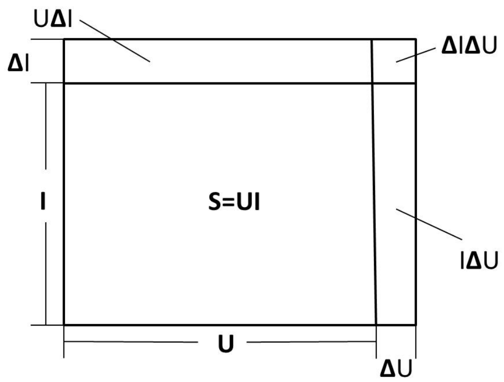
Error propagation

If a measurement result is
formed from multiple measured values, the individual errors associated with each measured value all affect the overall measurement result.

Estimating error in multiplication of measured values

Example: Measurement of apparent power.

The error propagation can be illustrated by the case of an apparent power measurement, S, which is the multiplication of the two measurement quantities voltage U and current I, S = U I. The measurement instrument in this example has a 0.5% error, or 0.5% of the input value limit, for the voltmeter and 1.0% for the ammeter.



The graphic helps to illustrate how the area ΔS, which results from the errors ΔU and ΔI, is:

$$\Delta S = U\Delta I + I\Delta U + \Delta I\Delta U$$

Where $S + \Delta S = (U + \Delta U)(I + \Delta I)$ follows our basic power calculation. If the errors are sufficiently small, then $\Delta U\Delta I$ can be neglected by comparison with the other summands. The relative error is computed by dividing by S = U I

$$\frac{\Delta S}{S} = \frac{\Delta U}{U} + \frac{\Delta I}{I}$$

This result indicates that multiplication of measurement values implies addition of the relative errors. This is of general validity for measurement quantities combined by multiplication. Now we wish to express the relative error in terms of the accuracy rating by means of the guaranteed error margins. If the size and sign of the systematic error are known, an incorrect result can be corrected.

Letting U_E and I_E be the input limits of the measurement channels, and Kl be the relative error, the errors ΔU and ΔI are expressed as:

$$\Delta U = \frac{Kl_U U_E}{100} \quad \text{and} \quad \Delta I = \frac{Kl_I I_E}{100}$$

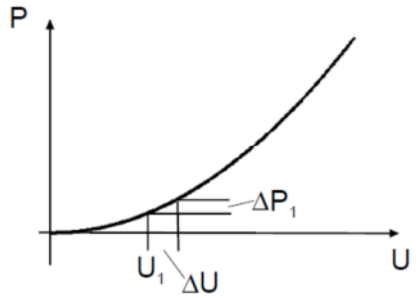
Thus, the relative error in the apparent power is calculated as:

$$\frac{\Delta S}{S} = \frac{Kl_U}{100}\frac{U_E}{U} + \frac{Kl_I}{100}\frac{I_E}{I}$$

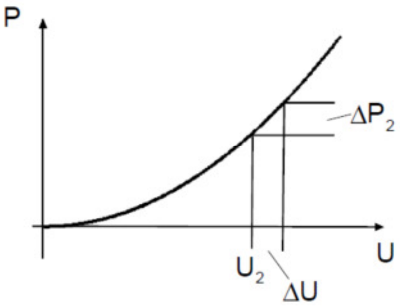
This result indicates that the minimum error amounts to the addition of accuracy ratings (addition of the %-values). The condition for this is that $U = U_E$ and $I = I_E$: the minimum error can only be achieved if the measured channels are operating at the peak of their range. So what applies to "old style" classical measurement devices continues to apply to modern measurement devices: to attain the minimum relative error, the fullest possible signal strength should be applied to the measurement channel. With measurement devices from imc, most measurement channels have an error limit of 0.1% of the input range limit, or a so-called Class 0.1, where the class specification (e.g., 0.1) typically signifies an error of the corresponding percentage (± 0.1%) of the input range end value.

A further example shall serve to clarify the mathematical background. Consider the determination of the voltage drop across a resistor whose resistance value of 1Ω, which we treat as absolutely accurate. The power is then calculated as $P=U^2/R$.

The power is thus P=(12V)2/1Ω=144 W. The voltage was measured as 12V, on a 20V range with an error of 1%. Thus, the possible error is computed as ΔU=0.2V. We seek the error in the power measurement ΔP. The graphs below show the error in the power ΔP as determined for different voltage values U_1 and U_2 but the same voltage error ΔU.



Measurement of the voltage U_1 with the error ΔU leads to the error ΔP_1.



Measurement of the voltage U_2 with identical ΔU leading to a much greater error ΔP_2.

As the figure above shows, the very same error size ΔU in both voltages U_1 and U_2 leads to completely different errors in the power ΔP_1 and ΔP_2. Evidently the size of the error in the power depends not only on the size of the measurement error ΔU, but also on the value of the voltage itself. This is in consequence of the quadratic relationship between the measured quantity U and the result quantity P. In other words, the error in the power depends on the slope of the curve at the measured value. The slope is expressed mathematically as:

$$\frac{dP}{dU} = \frac{2U}{R} \approx \frac{\Delta P}{\Delta U}$$

Generalizing from infinitesimally small magnitudes dP and dU to finite magnitudes ΔP and

ΔU, the increase of the function P from the equation above is seen to be calculated as:

$$\Delta P = \frac{dP}{dU}\Delta U \, .$$

In this particular case, the error is:

$$\Delta P = \frac{2U}{R}\Delta U \; or \; \Delta P = \frac{2\;12V}{1\Omega}0.2V = 4.8W$$

As is evident, the increase in a function's value can be computed by multiplying the slope with the change in the measured variable.

If n measurement quantities $x_1, x_2,...,x_n$ are combined to form a calculated result $y=y(x_1,x_2,...,x_n)$ and the systematic errors $\Delta x_1,...,\Delta x_n$ are relatively small, then the total differential can be used to determine the error Δy.

$$\Delta y = \frac{\partial y}{\partial x_1}\Delta x_1 + \frac{\partial y}{\partial x_2}\Delta x_2 + ... + \frac{\partial y}{\partial x_n}\Delta x_n$$

Thus, if multiple measurement quantities are present, the slope ratios:

$$\frac{\partial y}{\partial x}$$

of the individual measurement quantities are multiplied with the respective quantities relative error, and the resulting contributions of each individual quantity are added. The symbols ∂ in the equation above refer to partial derivatives. The method of partial differentiation is identical to differentiation with one variable, except that all other variables are regarded as constants.

The example of apparent power measurement presented above is thus computed as follows:

S = U I = S(U,I)

S corresponds to y
U corresponds to x_1
I corresponds to x_2

$$\Delta S = \frac{\partial S}{\partial U}\Delta U + \frac{\partial S}{\partial I}\Delta I$$

$$\Delta S = I\Delta U + U\Delta I$$

This result is identical with the result previously obtained graphically.

Certain and probable error margins

The total margin of error can be determined accordingly, if G_1 through G_n are the margins of error of the individual measurement quantities (corresponding to ΔU and ΔI in the example above):

$$\pm G_{yS} = \sum_{i=1}^{n} \left| \frac{\partial y}{\partial x_i} G_i \right|$$

G_{yS} is the result's error margin.

Since in practical terms it is improbable for the errors of all the quantities to coincide at the same (either the positive or negative) error margin, it is unlikely for the outer regions of the certain error margin to be reached. For this reason, the probable error margin G_{yW} is additionally defined.

$$\pm G_{yW} = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i} G_i \right)^2}$$

Error estimation for complex formulas

Example: Computing a drag coefficient.

As a final example, we will examine the drag coefficient C_W of a motor vehicle. In a wind tunnel, the air velocity v, air density ρ and resistance force F are measured. The frontal area A is optimized. The equation for computing the resistance force is:

$$F = \frac{\rho}{2} v^2 c_w A \,.$$

The c_W-value and its certain error margins are to be determined. With imc measurement instruments, the c_W-value can be calculated from the measured quantities in real time using the digital signal processor accessible through imc Online FAMOS. First, we solve for the c_W-value from its functional relationship with the force F. The result of this operation is:

$$c_W = \frac{2F}{\rho v^2 A}$$

Thus the maximum error for the c_W-value with the total differential is computed as:

$$\pm \Delta C_W = \left|\frac{\delta C_W}{\delta F}\Delta F\right| + \left|\frac{\delta C_W}{\delta \rho}\Delta \rho\right| + \left|\frac{\delta C_W}{\delta}\Delta V\right| + \left|\frac{\delta C_W}{\delta}\Delta A\right|$$

Given the functional relationship, the computation produces:

$$\pm \Delta C_W = \left|\frac{2}{\rho v^2 A}\Delta F\right| + \left|\frac{-2F}{\rho^2 v^2 A}\Delta \rho\right| + \left|\frac{-4F}{\rho v^3 A}\Delta V\right| + \left|\frac{-2F}{\rho v^2 A^2}\Delta A\right|$$

The following values were measured: F = 200 N; ρ = 1.2 kg/m3; v = 150 m/s; A = 400 cm^2.

The calculation of the nominal value for c_W, applying the identity 1N=1kg m/s², yields:

$$C_W = \frac{2F}{\rho v^2 A} = \frac{2 \cdot 200N}{1.2 \mathrm{kg/m^3} \left(\frac{150 \mathrm{m}}{\mathrm{s}}\right)^2 400 cm^2} = 0.3704$$

The missing maximum errors ΔF, Δρ, etc. for the measured physical quantities comprise the errors of the sensors and the measurement channel errors. The measurement of the force provides an occasion to further examine the error. The force sensor itself has a maximum error of 0.1% of the 250N input range, and it is connected to the imc CRONOS-PL measurement system. The maximum error occurring in the measurement channel in the 250N input range (associated with the full scale of the channel's capacity) is also rated at 0.1% of the input range's upper limit.

Since the sensor and measurement channel are multiplied with each other, the relative errors add (see the above example of determining the apparent power) and the maximum error in the force is 0.2% of 250 N, in other words 0.5 N. Correspondingly, the maximum errors of the other quantities are calculated as $\Delta\rho = 0.0025$ kg/m³; $\Delta v = 0.4$m/s; $\Delta A = 0.05$cm².

Calculation of the error yields:

$$\pm \Delta C_W = \left|\frac{2}{\rho v^2 A}\Delta F\right| + \left|\frac{-2F}{\rho^2 v^2 A}\Delta \rho\right| + \left|\frac{-4F}{\rho v^3 A}\Delta V\right| + \left|\frac{-2F}{\rho v^2 A^2}\Delta A\right|$$

for the maximum absolute error. Applying the values provided, we obtain the resulting error as:

$$\pm \Delta c_W = |0.0009259| + |0.0007716| + |0.0019753| + |0.0000463| = 0.003719$$

Dividing the error $\pm\Delta c_W$ by c_W, the maximum relative error based on the measurements is given by:

$$\pm\frac{\Delta C_W}{C_W} = \left|\frac{\Delta F}{F}\right| + \left|\frac{\Delta\rho}{\rho}\right| + \left|2\frac{\Delta V}{V}\right| + \left|\frac{\Delta A}{A}\right| = 0.01004 \text{ or approx. } 1.0\%$$

Probable and certain margins of error

In this example, although the relative error in the individual measurement channels (in relation to their input ranges) is only 0.1% or better, the margin of error for the calculated measurement result is no better than 1%. And this even assumes that no additional error factor is incurred as a result of processing with imc Online FAMOS, which in this case is indeed fair to assume. The relative error in the velocity impacts the overall relative error twice as strongly due to its quadratic relationship to the result. Another insight to be gained is that the measurement channels should carry signals as near as possible to their maximum capacity level, in order to keep the relative error low. The calculated error represents a certain (guaranteed) margin of error. As indicated above, the probable margin of error can also be computed, which in our cases comes to:

$$\pm\sqrt{(0.0009259)^2 + (0.0007716)^2 + (0.0019753)^2 + (0.0000463)^2} = 0.00231$$

corresponding to a relative error of 0.63%. The advantage of knowing this specification of the error is that it approximates the actual error level much better than the certain error margin of 1%. The drawback is, however, that it is not possible to state the probability that the error is no greater than 0.63%. To be on the safe side, the certain error margin must be calculated.

Additional information:

imc Test & Measurement GmbH
Voltastr. 5
13355 Berlin, Germany

Telephone: +49 (0)30-46 7090-0
Fax: +49 (0)30-46 31 576
E-mail: hotline@imc-tm.de
Internet: http://www.imc-tm.com

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